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LETTER TO THE EDITOR

The one-dimensional map $1 - Cx^{2\mu}$ in the large-μ limit

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Abstract. We show that for period doubling bifurcations in the map $1 - Cx^{2\mu}$, in the limit $\mu \gg 1$, C_∞ (the limit point of the bifurcations) and the universal exponents α and δ have non-analytic behaviour in μ.

The onset of period doubling chaos has been studied extensively (Feigenbaum 1978, Collet and Eckmann 1980, Hu 1982) in one-dimensional maps of the form $x_{n+1} = f(x_n) = 1 - Cx_n^{2\mu}$ for $1 < 2\mu < \infty$ motivated by the universal behaviour observed for the $\mu = 1$ case. Various analytic approximations have been studied by Derrida *et al* (1979) and more recently by Hauser *et al* (1984). For 2μ close to unity, a perturbation theory for calculating C_∞ (the limit point for the successive period doubling bifurcations) and the universal constants α and δ has been devised by Derrida *et al* (1979). From the results of their different renormalisation group transformations, Hauser *et al* have made some conjectures regarding the high-μ ($\mu \gg 1$) behaviour of the above quantities. Here we study specifically the high-μ behaviour of C_∞, α and δ, and show that *their dependence on μ is non-analytic for $\mu \gg 1$.*

We proceed directly by finding the value C_N of C for the superstability of the 2^N cycle. This is the value of C for which 0 is a stable fixed point of $f^{2^N}(x)$. Since $f^{2^1}(0) = 1 - C_1$, we immediately see that C_N is to be obtained as the root of

$$0 = f^{2^N}(0) = f^{2^N-2}(f^2(0)) = f^{2^N-2}(1 - C_N). \tag{1}$$

We make use of the fact that C_N will be close to 2 and hence can be written as

$$C_N = 2 - \phi_N/2\mu \tag{2}$$

where $\phi_N/2\mu \ll 1$. Making repeated use of the estimate

$$x^{1/\mu} \simeq 1 + \frac{1}{\mu}\ln x \tag{3}$$

for $\mu \gg 1$, we can express ϕ_N as the solution of

$$\phi_N = \ln\left[\frac{2\mu C_N}{\ln\left[\frac{2\mu C_N}{\left[\ \ddots\ \frac{}{\ln\left(\frac{2\mu C_N}{\ln C_N}\right)}\right]}\right]}\right] \tag{4}$$

where the number of log factor is $2^N - 2$. The derivation of the above equation can be seen as follows. We take $N = 2$ and for C_2 find from equation (1),

$$0 = f^{(2)}(1 - C_2) = 1 - C_2\{f(1 - C_2)\}^{2\mu}$$
$$= 1 - C_2\{1 - C_2(1 - C_2)^{2\mu}\}^{2\mu} \tag{5}$$

whence,

$$1 - C_2(1 - C_2)^{2\mu} = \left(\frac{1}{C_2}\right)^{\mu/2} \simeq 1 - \frac{1}{2\mu}\ln C_2 \tag{6}$$

making use of equation (3). This leads to

$$C_2 - 1 = \left(\frac{1}{2\mu C_2}\ln C_2\right)^{\mu/2} \simeq 1 + \frac{1}{2\mu}\ln(\tfrac{1}{2}\mu C_2 \ln C_2)$$

or

$$\phi_2 = \frac{1}{2\mu}\ln\left(\frac{2\mu C_2}{\ln C_2}\right) \tag{7}$$

where $C_2 = 2 - \phi_2/2\mu$. Carrying out this procedure 2^{N-2} times leads to equation (4).

The accumulation point of the C_N is the accumulation point of the period doubling bifurcations and being $C_\infty = \text{Lim}_{N\to\infty} C_N$ can be written as

$$C_\infty = 2 - \phi/2\mu \tag{8}$$

where

$$\phi = \ln\left(\frac{4\mu}{\phi}\right) \tag{9}$$

is the large-μ fixed point of the recursion relation for ϕ_N which follows from equation (4). In table 1 we show the comparison between equations (8) and (9) and the exact (numerically determined) C_∞. The agreement for large μ is good to within 1 in 10^3. The non-analytic behaviour in μ is evident from equations (8) and (9) which would give the lowest order approximation to C_∞ as

$$C_\infty \simeq 2 - \ln 4\mu/2\mu. \tag{10}$$

The $\mu^{-1}\ln\mu$ term is the source of the non-analytic behaviour of C_∞ and, in turn, of the universal constants α and δ.

Table 1.

2μ	$C_\infty = 2 - \dfrac{\phi}{2\mu}$	C_∞ exact	$\alpha = \left(1 - \dfrac{\phi}{2\mu}\right)^{-1}$	$\delta = \alpha^{2\mu} - \alpha$	δ exact
6	1.6895	1.683 26	1.45	7.85	9.2997
10	1.7795	1.772 64	1.287	11.03	12.376
30	1.9000	1.896 77†	1.111	22.48	20.5271†
100	1.9607	1.959 26	1.045	54	29.0510†

† These values are the result of the most accurate version of the renormalisation group calculation of Hauser *et al.*

We turn now to a determination of α. We know from the standard definitions that

$$\alpha = \lim_{N\to\infty} \{-[f^{2^N}(0)]^{-1/N}\}. \tag{11}$$

The lowest order approximation ($N = 1$) gives

$$\alpha = (C_\infty - 1)^{-1} = (1 - \phi/2\mu)^{-1} \simeq 1 + (\phi/2\mu). \tag{12}$$

Once again we see that as μ increases α approaches 1 in a non-analytic fashion because of the logarithmic terms in ϕ. The convergence of equation (12) for large μ is swift, so much so that $N = 1$ assures 1% accuracy. To improve on that we note that if we define $(-\alpha)^{-N} = f^{2^N}(0)$, then α_N satisfies a recursion relation

$$(-\alpha_{N+1})^{-(N+1)} = f^{2^N}((1/-\alpha_N)^N) \tag{13}$$

and α is obtained as a fixed point of this relation as $N \to \infty$. For $N = 1$, the fixed point relation is

$$\begin{aligned}\alpha^{-2} &= f^2(1/\alpha)\\ &= 1 - C_\infty\left(1 - \frac{C_\infty}{\alpha^2\mu}\right)^{2\mu}\end{aligned} \tag{14}$$

leading to the same asymptotic solution as that given in equation (12). Thus the lowest order prescription for α and the lowest order fixed point value of α lead to the same result showing the relative insensitivity of α to the different techniques of evaluation for $\mu \gg 1$. At this point we note that Eckmann and Wittwer (1985) have determined the asymptotic form ($\mu \to \infty$)

$$\alpha \simeq 1 + \ln(\alpha^{2\mu})/\mu \tag{15}$$

with a computer assisted proof. They now assume on the basis of numerical evidence that $\lim \alpha^{2\mu}$ is finite and nearly equal to 30 for $\mu \to \infty$. This leads to

$$\alpha \simeq 1 - \ln(1033)/\mu. \tag{16}$$

In the form of equation (15), their asymptotic form is consistent with our equations (12) and (14).

Finally we determine δ. For $\mu \gg 1$, as noted by Feigenbaum (1978), δ is given accurately by the prescription

$$\delta = \alpha^{2\mu} - \alpha. \tag{17}$$

Using equation (12) for α and taking the large-μ limit, we find that

$$\delta = e^\phi - 1 \tag{18}$$

implying an asymptotic form

$$\delta \simeq \frac{4\mu}{\ln 4\mu}. \tag{19}$$

The values of δ computed from equation (18) are also shown in table 1. We note that the agreement with numerical values (Derrida *et al* 1979, Hauser *et al* 1984), although good, is less impressive than that for C_∞. The reason lies in the fact that the form of equation (18) magnifies the error in the determination of δ from α. It should be emphasised that while the large-μ form of C_∞ is exact, those for α and δ are not. Thus while equation (12) is a good analytic estimate of α for $2\mu \gg 1$, taking the limit of the sequence in equation (11) might alter the asymptotic form.

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